MATH 2850: CONVOLUTION

DEFINITION: $(f * g)(t) = \int_0^t f(u)g(t-u)\,du$

NOTE: $(f * 1)(t) = \int_0^t f(u)\,du$

VISUALIZATION?

EXAMPLE: Use the integral definition of convolution to find the following:

- $t * \sin(t)$

$$t * \sin(t) = t - \sin(t)$$

- $\sin(t) * t$

$$\sin(t) * t = t - \sin(t)$$

PROPERTIES OF CONVOLUTION:

- **COMMUTATIVE:** $f * g = g * f$
- **ASSOCIATIVE:** $(f * g) * h = f * (g * h)$
- **DISTRIBUTIVE:** $f * (g + h) = f * g + f * h$
- **SCALARS FLOAT:** $c(f * g) = (cf) * g = f * (cg)$
- **DERIVATIVE:** $D_t\,[f(t) * g(t)] = f(t)g(0) + f(t) * g'(t)$

THEOREM: Convolution and Laplace Transforms:

- $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\} = F(s)G(s)$
- $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$
- $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \mathcal{L}\{f * 1\} = \dfrac{F(s)}{s}$
- $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

EXAMPLE: Find $\mathcal{L}^{-1}\left\{\dfrac{8}{s^3 + 4s}\right\}$ two ways: first, using partial fractions; second, using convolution.

Using partial fractions: $\mathcal{L}^{-1}\left\{\dfrac{8}{s^3 + 4s}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s} - \dfrac{2s}{s^2 + 4}\right\} = 2 - 2\cos(2t)$

Using convolution: $\mathcal{L}^{-1}\left\{\dfrac{8}{s^3 + 4s}\right\} = \mathcal{L}^{-1}\left\{\dfrac{4}{s}\right\}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2 + 4}\right\} = 4 * \sin(2t) = 2 - 2\cos(2t)$

EXAMPLE: Find the following using Laplace Transforms:

- $t * e^t$

Ans: $t * e^t = e^t - t - 1$

- $\displaystyle\int_0^t 2e^{-u}\cos(t-u)\,du$

Ans: $\displaystyle\int_0^t 2e^{-u}\cos(t-u)\,du = 2e^{-t} * \cos(t) = \sin(t) + \cos(t) - e^{-t}$

EXAMPLE: Solve the integral equation: $f(t) = 2t - \int_0^t \sin(u) f(t-u)\, du$

Ans: $f(t) = t + \frac{1}{\sqrt{2}} \sin(\sqrt{2}\, t)$

EXAMPLE: Solve the following IVP in terms of convolutions: $y'' + 25y = f(t)$, $y(0) = 1$, $y'(0) = -1$.

Ans: $y(t) = \cos(5t) - \frac{1}{5} \sin(5t) + \frac{1}{5} \sin(5t) * f(t)$

EXAMPLE: Solve the following IVP in terms of convolutions: $ay'' + by' + cy = f(t)$, $y(0) = k_0$ and $y'(0) = k_1$.

HOMEWORK: Section 8.6: Pg. 449: 1 - 5 'odd letters' (a, c, e, etc.)